Recursion relations for gauge theory amplitudes with massive vector bosons and fermions

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ABSTRACT: We apply the on-shell tree-level recursion relations of Britto, Cachazo, Feng and Witten to a variety of processes involving internal and external massive particles with spin. We show how to construct multi-vector boson currents where one or more off-shell vector bosons couples to a quark pair and number of gluons. We give compact results for single vector boson currents with up to six partons and double vector boson currents with up to four partons for all helicity combinations. We also provide expressions for single vector boson currents with a quark pair and an arbitrary number of gluons for some specific helicity configurations. Finally, we show how to generalise the recursion relations to handle massive particles with spin on internal lines using $gg \to t\bar{t}$ as an example.

Contents

1. Introduction

Powerful new formalisms have been recently developed for the calculation of on-shell quantities such as scattering amplitudes in gauge theories. Most striking are the MHV rules of ref. [1] and the Britto-Cachazo-Feng-Witten (BCFW) recursion relations of refs. [2, 3]. Applications of these new formalisms together with the classic unitarity based approach of refs. [4, 5] have led to a dramatic progress in calculations of amplitudes.

The 'MHV rules', originally proposed in [1], led to simple and compact calculations of tree level amplitudes [6–12]. The method has also been generalised to include external Higgs bosons and massive vector bosons [13–15] as well for investigating collinear factorisation [16, 17]. Another recent and novel approach to the computation of amplitudes with massive quarks [18] discusses an MHV-like formalism that has the virtue of a direct connection with the QCD Lagrangian.

The on-shell BCFW tree-level recursion relations were first formulated for massless particles [2, 3, 19–22]. The proof of the new recursion relations [3] relies on simple analytic properties of the amplitudes and have recently been generalised in three ways. First, the massless on-shell recursion relations in gauge theory were extended to include gravity [23, 24]. Second, a new version of recursion relations was developed in refs. [25, 26] to calculate all finite one-loop amplitudes in non-supersymmetric QCD. Ref. [27] further generalised

this approach to compute the cut-nonconstructible parts of divergent one-loop gluonic amplitudes. In a third development, ref. [28] generalised the BCFW recursion relations to include massive scalar particles at tree level. The goal of this paper is to further extend the recursion relations to include massive particles with spin such as vector bosons or massive quarks in a natural way. We will show that the recursion relations lead to new compact formulae for scattering amplitudes that are somewhat simpler than the previous calculations of Berends, Giele and Kuijf [29, 30].

Much progress has also been made at one loop level. The MHV rules have been successfully applied to compute supersymmetric amplitudes [31 – 37] as well as new calculations using improved unitarity methods [38 – 48]. The new formalism has been largely inspired by Witten's idea of a duality between supersymmetric Yang-Mills and a topological string theory on a twistor space [49]. For reviews of recent developments see [50, 51].

Our paper is organised as follows. In section 2 we review the recursive formulation of gauge theory amplitudes. Section 3 defines our model and reviews colour ordering of the amplitudes. We provide new compact results for amplitudes with up to six partons and a single vector boson in section 4 and with up to four partons and two vector bosons in section 5. New compact expressions for n-point NMHV currents with a single vector boson are also given in section 4. Finally we demonstrate how to use recursion relations including propagating massive particles with spin in section 6. Our findings are briefly summarized in section 7. An appendix stating our spinor conventions is also included.

2. On-shell recursion relations

Here we give a brief summary of the on-shell recursion relations [2] for tree-level amplitudes. These recursion relations follow from general quantum field theory arguments [3] and we will present them in the form [28] which naturally incorporates massless and massive particles. We consider the colour-ordered partial amplitudes $\mathcal{A} = \mathcal{A}(p_1, \ldots, p_n)$, in which the coloured particles come in a definite cyclic order $1, 2, \ldots, n$. These amplitudes are obtained by stripping away the colour factors from the full amplitude, hence, they depend on the kinematic variables, momenta and helicities, p_k and h_k only.

The formalism relies on choosing two particles in the amplitudes to be shifted by a complex vector to be specified below. We label these marked particles i and j. The on-shell recursion relation for a tree-level amplitude takes the form:

$$\mathcal{A}_n(p_1, \ldots, p_n) = \sum_{\text{partitions}} \sum_h \mathcal{A}_L(p_r, \ldots, \widehat{p}_i, \ldots, p_s, -\widehat{P}^h) \frac{1}{P^2 - m_P^2} \times$$
$$\times \mathcal{A}_R(\widehat{P}^{-h}, p_{s+1}, \ldots, \widehat{p}_j, \ldots, p_{r-1}) , \qquad (2.1)$$

where $P = p_r + \ldots + p_i + \ldots + p_s$ and the "hatted" quantities are the shifted on-shell momenta. The summation in (2.1) is over all partitions of n external particles between the smaller amplitudes on the left, $\mathcal{A}_L$, and on the right, $\mathcal{A}_R$, such that p_i is on the left, and p_j is on the right, and also over the helicities, h, of the intermediate state.

In this paper we will always choose both marked particles i and j to be massless. We shift two massless momenta[1] $p_i = |i\rangle|i]$ and $p_j = |j\rangle|j]$ of the marked particles by $\eta = |j\rangle|i]$, such that the shifted momenta are

$$\widehat{p}_i = p_i + z|j\rangle|i] , \qquad \widehat{p}_i^2 = 0 = p_i^2 , \tag{2.2}$$

$$\widehat{p}_j = p_j - z|j\rangle|i] , \qquad \widehat{p}_j^2 = 0 = p_j^2 , \tag{2.3}$$

$$\widehat{P} = P + z|j\rangle|i] , \qquad \widehat{P}^2 = m_P^2 . \tag{2.4}$$

Here m_P is the mass of the particle on the internal line. Equation (2.4) determines the variable z as

$$z = -\frac{P^2 - m_p^2}{2P \cdot \eta} = -\frac{P^2 - m_p^2}{\langle j|P|i]} . \tag{2.5}$$

For the particles i, j momentum shifts above are equivalent to shifting the spinors as follows:

$$|\widehat{i}] = |i] , \qquad |\widehat{i}\rangle = |i\rangle + z|j\rangle , \tag{2.6}$$

$$|\widehat{j}\rangle = |j\rangle , \qquad |\widehat{j}] = |j] - z|i] . \tag{2.7}$$

A direct proof of (2.1) was given in reference [3] based on the analyticity structure of the meromorphic tree-level amplitude $\mathcal{A}_n(z)$ in the complex z-plane. For equation (2.1) to hold it is essential that $\mathcal{A}_n(z)$ should have no poles at infinity. In general it is known that whether or not these "boundary" contributions are present (i.e. whether or not (2.1) is valid) can depend on the choice of the marked particles i and j [3, 21, 28]. For the classes of models considered in this paper we will be always marking only massless particles, and in such a way that the helicities of the marked particles can take the values,

$$(h_i, h_j) = (+, -) , (+, +) , (-, -) \tag{2.8}$$

but not $(h_i, h_j) = (-, +)$. In addition to this, if the two marked particles are a quark and an anti-quark of the same flavour, they should not be adjacent. Finally, for adjacent quarks and gluons of equal helicity cases we make the choices (j_q^+, i_g^+) or (i_q^-, j_g^-).

3. Massive vector bosons

The principal goal of this paper is to apply on-shell recursion relations to tree-level amplitudes involving massive or off-shell particles with spin coupled to massless particles. To begin with we will consider massive vector bosons coupled to massless gauge fields and fermions. Essentially we consider a generic theory with a non-Abelian gauge group being a product $G_1 \times G_2$, where G_1 is unbroken, and G_2 is broken by the Higgs mechanism. Gauge fields of the G_1 group are massless and the gauge fields of broken group G_2 are massive or off-shell vector bosons V. Two gauge groups are coupled to each other via fermions which are charged under both groups. We will use the 'colour decomposition' representation for

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[1]Helicity spinors $|i\rangle$, $|i]$ are defined in the Appendix along with our conventions for forming the inner product.

scattering amplitudes with respect to both groups, and hence the colour-stripped amplitudes will be purely kinematic quantities which will not depend on the choice of G_1 and G_2 nor on the representations for the matter fermions.

This set-up is rather general, and in particular it incorporates the elements of the Standard Model. In this case G_1 is $SU(3)$ and the corresponding gauge fields are gluons g; the gauge fields of the (partially) broken group, $G_2 = SU(2) \times U(1)$, are massive or off-shell vector bosons $W^\pm$, Z^0 and γ^*. The fermions can be taken to be (anti-)quarks, $\bar{q}$, q, transforming in the (anti-)fundamental representations of both groups. Even in the general case, we will continue denoting massless gauge fields as gluons, and fermions as quarks. Massive vector bosons will be denoted as V's.

The quantities we want to consider are the G_1- and G_2-colour-stripped purely kinematic tree-level amplitudes

$$S_{\mu_1 \dots \mu_m}(1_q, 2, 3, \dots, n-1, n_{\bar{q}}). \tag{3.1}$$

These are the $(m+n)$-point amplitudes with m massive vector bosons $V_{\mu_1}, \dots V_{\mu_m}$ coupled to n massless partons. More specifically, we consider the case of a single quark-antiquark pair[2] denoted in (3.1) as $1_q, n_{\bar{q}}$ and $n-2$ gluons labelled $2, 3, \dots, n-2$.

The group-theoretical dependence in the amplitudes can be easily restored in the usual way, se for example [52]. For the case of fundamental fermions the amplitude (3.1) is multiplied by $(T^{a_2} \dots T^{a_{n-1}})_{i_1 i_n}$ and by $(T^{b_1} \dots T^{b_m})_{k_q k_{\bar{q}}}$. Then it is summed over all permutations of $a_2, \dots, a_{n-1}$ and over all permutations of $b_1, \dots, b_m$. Here T^a and T^b are the generators of the G_1 and the G_2 groups respectively.

The physical states corresponding to all massless particles in amplitudes (3.1) will always be represented in the helicity basis, e.g. $1_q^-, 2^+, 3^- \dots, (n-1)^+, n_{\bar{q}^-}$. At the same time, for the massive (off-shell) vectors $V_{\mu_1}, \dots V_{\mu_m}$ we will always choose to not multiply them by external wave functions, and instead of helicities or polarisations they will be characterised by their Lorentz indices $\mu_1, \dots, \mu_m$. Thus, the amplitudes $S_{\mu_1 \dots \mu_m}$ are the multi-vector boson currents.

Working with multi-currents (3.1) will first of all facilitate our calculation: single vector currents will be used in calculations of double currents and so on, as will be seen in section **5**. Furthermore, multi-currents can be easily used to calculate full physical amplitudes which include the decay of the massive (off-shell) vector bosons into light stable states. This is achieved by contracting each Lorentz index μ in (3.1) with the current L^μ describing the relevant decay mode of each vector boson V_μ.

In the Standard Model, for example, one can consider decays of unstable vector bosons into a fermion-antifermion (lepton or quark) pair, so that for virtual photon γ^* or for $V = W^\pm, Z$ boson decay we have:

$$L^\mu_{\gamma^*} = -e^2 Q_f Q_q \frac{\bar{u}(p_f)\gamma_\mu u(p_{\bar{f}})}{P^2_{\gamma^*}}$$

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[2]One cannot have less than one $q\bar{q}$-pair in amplitudes coupling V's to g's at tree level. Amplitudes with more than one $q\bar{q}$-pair will not be considered in this paper, but they can be calculated in a similar way.

$$L_V^\mu = -e^2 \frac{v_{V;H}^{f\bar{f}} v_{V;H}^{q\bar{q}} \bar{u}(p_f)\gamma_\mu u(p_{\bar{f}})}{(P_V^2 - M_V^2 + i\Gamma_V M_V)}. \tag{3.2}$$

Here the couplings $v_{V;H}$ for V either W or Z bosons with either left (L) or right (R) handed polarisations are given by

$$v_{Z;R}^{f\bar{f}} = -Q_f \frac{\sin\theta_w}{\cos\theta_w} \ , \qquad v_{Z;L}^{f\bar{f}} = -Q_f \frac{I_{w,f}^3 - \sin^2\theta_w}{\sin\theta_w \cos\theta_w} \ , \qquad v_{W;L}^{\nu_i \bar{l}_j} = \frac{1}{\sqrt{2}\sin\theta_w}\delta_{ij} \ ,$$

$$v_{W;L}^{l_i \bar{\nu}_j} = \frac{1}{\sqrt{2}\sin\theta_w}\delta_{ij} \ , \qquad v_{W;L}^{u_i \bar{d}_j} = \frac{1}{\sqrt{2}\sin\theta_w}U_{ji}^\dagger \ , \qquad v_{W;L}^{d_i \bar{u}_j} = \frac{1}{\sqrt{2}\sin\theta_w}U_{ji} \ . \tag{3.3}$$

and all others zero. U_{ij} is the CKM mixing matrix and the rest of notation is standard.

4. Single vector boson currents

The single vector boson currents were previously calculated by Berends, Giele and Kuijf [30] using the recursive technique based on iterations of classical equations of motion [29]. More recently these single vector currents were also discussed and derived in [15] using a combination of Berends-Giele recursion relations and the MHV rules of [1]. Here we will employ the BCFW on-shell recursion relations of section 2 to derive slightly more compact expressions as well as new results for n-parton single currents for some specific helicity arrangements of partons. As mentioned earlier, single off-shell currents S_μ are not only interesting on their own right, more significantly, they play an important part in a recursive construction of currents with two and more vector bosons as will be explained in the next section. We note that a similar observation has also been made in [15].

We now proceed to construct the single vector boson currents,

$$S_\mu(1_q^\lambda, 2^{h_2}, \ldots, (n-1)^{h_{n-1}}, n_{\bar{q}}^{-\lambda}). \tag{4.1}$$

The on-shell recursion relations construct amplitudes from on-shell amplitudes with fewer particles. Assuming that one can always avoid contributions at $z \to \infty$ and using the recursion relation $n-3$ times gives a representation of the n-point amplitude entirely in terms of the 3-point vertices. Hence, 3-point vertices are the building blocks of all larger amplitudes and hence the on-shell recursion relations reduce the task of computing general amplitudes to the computation of all 3-point vertices. This is indeed the case in theories with massless vectors coupled to fermions and also to massless and massive scalars[3] [2, 3, 28].

The three-gluon and the quark-gluon-antiquark primitive amplitudes are given by the standard MHV and $\overline{\text{MHV}}$ expressions:

$$\mathcal{A}_3(1^-, 2^-, 3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} \ , \qquad \mathcal{A}_3(1^+, 2^+, 3^-) = -\frac{[12]^3}{[23][31]} \ , \tag{4.2}$$

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[3]This is however not always the case in theories involving scalar self-interactions. The 4-point vertex corresponding to a ϕ^4 interaction clearly cannot be reduced to 3-point vertices. Recursion relation cannot be applied to this vertex since it is a constant. Hence the corresponding amplitude $A_4(z)$ does not depend on z, and this necessarily leads to a non-vanishing contribution at $z \to \infty$. We thank George Georgiou for pointing this out to us.

$$\mathcal{A}_3(1_q^-, 2^-, 3_{\bar{q}}^+) \ = \ \frac{\langle 12 \rangle^2}{\langle 13 \rangle} \ , \qquad \mathcal{A}_3(1_q^-, 2^+, 3_{\bar{q}}^+) \ = \ -\frac{[23]^2}{[13]} \ , \qquad (4.3)$$

$$\mathcal{A}_3(1_q^+, 2^-, 3_{\bar{q}}^-) \ = \ \frac{\langle 23 \rangle^2}{\langle 13 \rangle} \ , \qquad \mathcal{A}_3(1_q^+, 2^+, 3_{\bar{q}}^-) \ = \ -\frac{[12]^2}{[13]} \ . \qquad (4.4)$$

Here as always, all momenta k_i are assumed to be complex which ensures that these 3-point amplitudes do not vanish on-shell [2, 3].

In addition we need to introduce two new primitive vertices for an off-shell vector boson coupled to a $q\bar{q}$ pair. They are derived directly from the Feynman rules:

$$S_\mu(1_q^-, 2_{\bar{q}}^+) \ = \ \langle 1|\sigma_\mu|2] \equiv [2|\bar{\sigma}_\mu|1\rangle \ ,$$
$$S_\mu(1_q^+, 2_{\bar{q}}^-) \ = \ [1|\bar{\sigma}_\mu|2\rangle \equiv \langle 2|\sigma_\mu|1] \ . \qquad (4.5)$$

Here, $\sigma_{\mu\,\alpha\dot{\alpha}}$ and $\bar{\sigma}_\mu^{\dot{\alpha}\alpha}$ are the standard four Pauli matrices. Our conventions for spinor contractions are summarised in the Appendix.

Action of parity symmetry reduces the number of independent currents. Parity transformation is simply the complex conjugation in terms of $S_{\alpha\dot{\beta}} := S_\mu \sigma_{\alpha\dot{\beta}}^\mu$,

$$S_{\alpha\dot{\beta}}(1_q^\lambda, 2^{h_2}, \ldots, (n-1)^{h_{n-1}}, n_{\bar{q}}^{-\lambda}) \ = \ \left(S_{\beta\dot{\alpha}}(1_q^{-\lambda}, 2^{-h_2}, \ldots, (n-1)^{-h_{n-1}}, n_{\bar{q}}^\lambda) \right)^* . \qquad (4.6)$$

This formula is generalised to multi-vector boson currents in an obvious way:

$$S_{\alpha_1\dot{\beta}_1\ldots\alpha_m\dot{\beta}_m} \ = \ (S_{\beta_1\dot{\alpha}_1\ldots\beta_m\dot{\alpha}_m})^* . \qquad (4.7)$$

4.1 Single currents with $n = 4, 5, 6$ partons

In the four-parton case the recursion relation for S_μ reduces to,

$$S_\mu(1_q^-, 2^\pm, 3_{\bar{q}}^+) = S_\mu(\hat{1}_q^-, -\hat{P}_{\bar{q}}^+) \frac{1}{s_{23}} \mathcal{A}(\hat{P}_q^-, \hat{2}^\pm, 3_{\bar{q}}^+). \qquad (4.8)$$

We now have to choose which of the marked particles, $\hat{1}$, $\hat{2}$, is $\hat{i}$ and which is $\hat{j}$. In order to avoid boundary contributions according to (2.8) we must choose $i = 2$ for the 2^+ helicity and $j = 2$ for 2^- helicity. This results in,

$$S_\mu(1_q^-, 2^+, 3_{\bar{q}}^+) = -\frac{\langle 1|\sigma_\mu P_V|1\rangle}{\langle 12 \rangle \langle 23 \rangle} \ , \qquad (4.9)$$

$$S_\mu(1_q^-, 2^-, 3_{\bar{q}}^+) = \frac{[3|P_V \sigma_\mu|3]}{[12][23]} \ , \qquad (4.10)$$

where P_V is the momentum of the vector boson. Momentum conservation implies $P_V = -p_1 - p_2 - p_3$. The two remaining helicity configurations can be obtained by parity transformations.

Figure 1 shows the decomposition of the five-parton current. Just as in the four-parton case here we mark the quark $\hat{1}$ and adjacent gluon $\hat{2}$,

$$S_\mu(1_q^-, 2^+, 3^+, 4_{\bar{q}}^+) = \frac{\langle 1|\sigma_\mu P_V|1\rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle} \ , \qquad (4.11)$$
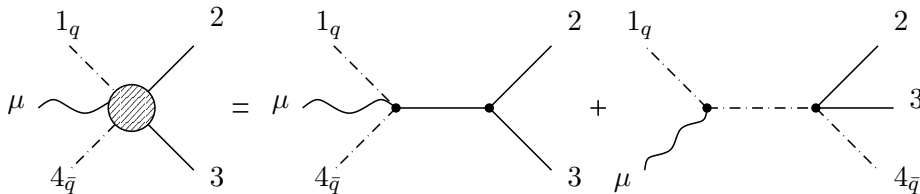
Figure 1: Decomposition of the five-parton amplitude after applying the recursion relation.

$$S_\mu(1_q^-, 2^-, 3^+, 4_{\bar{q}}^+) = \frac{[13][3|(1+2)P_V\bar{\sigma}_\mu(1+2)|3]}{s_{123}[12][23]\langle 4|2+3|1]}$$
$$+ \frac{\langle 24\rangle\langle 2|(3+4)\bar{\sigma}_\mu P_V(3+4)|2\rangle}{s_{234}\langle 23\rangle\langle 34\rangle\langle 4|2+3|1]} , \tag{4.12}$$

$$S_\mu(1_q^-, 2^+, 3^-, 4_{\bar{q}}^+) = \frac{[24]^3\langle 1|\sigma_\mu P_V|1\rangle}{s_{234}[23][34]\langle 1|2+3|4]}$$
$$+ \frac{\langle 13\rangle^3[4|P_V\sigma_\mu|4]}{s_{123}\langle 12\rangle\langle 23\rangle\langle 1|2+3|4]} . \tag{4.13}$$

The $S_\mu(+; ++; -)$ configuration can be obtained from eq. (4.11) by using the "line reversal" symmetry,

$$S_\mu(1_q^\lambda, 2^{h_2}, \ldots, n-1^{h_{n-1}}, n_{\bar{q}}^{-\lambda}) = (-1)^{n+1} S_\mu(n_q^{-\lambda}, n-1^{h_{n-1}}, \ldots, 2^{h_2}, 1_{\bar{q}}^\lambda) . \tag{4.14}$$

The other 4 helicity amplitudes can then by obtained via parity transformations. Notice that if we use both (4.6) and (4.14) we can also relate (4.10) and (4.9).

The 6-point amplitudes can be computed in much the same way. We choose to mark massless particles in such a way as to generate the most compact analytic expression. The following amplitude was computed using $i = 2$ and $j = 1$:

$$S_\mu(1_q^-, 2^+, 3^+, 4^+, 5_{\bar{q}}^+) = -\frac{\langle 1|\sigma_\mu P_V|1\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle} . \tag{4.15}$$

The following expressions was derived using $i = 3$ and $j = 4$:

$$S_\mu(1_q^-, 2^-, 3^+, 4^+, 5_{\bar{q}}^+) = -\frac{\langle 25\rangle\langle 2|(3+4+5)P_V\sigma_\mu(3+4+5)|2\rangle}{s_{2345}\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 5|P_V|1]}$$
$$+ \frac{[13][3|(1+2)\sigma_\mu P_V(1+2)|3]}{s_{123}\langle 45\rangle[12][23]\langle 4|2+3|1]}$$
$$+ \frac{\langle 2|3+4|1]\langle 2|(3+4)(1+2+3+4)\sigma_\mu P_V(1+2+3+4)(3+4)|2\rangle}{s_{1234}s_{234}\langle 23\rangle\langle 34\rangle\langle 5|P_V|1]\langle 4|2+3|1]} . \tag{4.16}$$

For the choice $i = 2$ and $j = 1$ we find:

$$S_\mu(1_q^-, 2^+, 3^-, 4^+, 5_{\bar{q}}^+) = -\frac{\langle 13\rangle^3\langle 35\rangle\langle 3|(4+5)P_V\sigma_\mu(4+5)|3\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 5|P_V(1+2)|3]\langle 3|(4+5)P_V|1]}$$
$$+ \frac{\langle 35\rangle\langle 3|4+5|2]^3\langle 1|\sigma_\mu P_V|1\rangle}{s_{2345}s_{345}\langle 34\rangle\langle 45\rangle\langle 3|(4+5)P_V|1]\langle 5|3+4|2]}$$

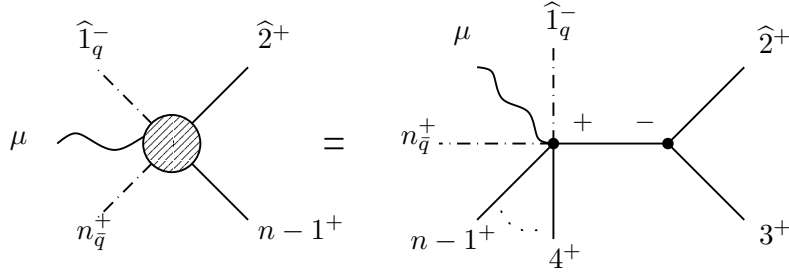**Figure 2:** Decomposition of the n-point vector boson current with $n-2$ positive helicity gluons.

$$-\frac{\langle 13\rangle^3\langle 3|1+2|4][4|(1+2+3)\sigma_\mu P_V(1+2+3)|4]}{s_{1234}s_{123}\langle 12\rangle\langle 23\rangle\langle 5|P_V(1+2)|3\rangle\langle 1|2+3|4]}$$
$$-\frac{[24]^4\langle 1|\sigma_\mu P_V|1\rangle}{s_{234}\langle 5|3+4|2]\langle 1|2+3|4][23][34]}\,. \tag{4.17}$$

Finally using $i=3$ and $j=2$ we find,

$$S_\mu(1_q^-,2^+,3^+,4^-,5_{\bar q}^+) = \frac{\langle 14\rangle^3[5|P_V\sigma_\mu|5]}{s_{1234}\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 1|P_V|5]}$$
$$-\frac{[35]^3\langle 1|\sigma_\mu P_V|1\rangle}{s_{345}\langle 12\rangle\langle 2|3+4|5][34][45]}$$
$$+\frac{\langle 4|2+3|5]^3\langle 1|\sigma_\mu P_V|1\rangle}{s_{234}s_{2345}\langle 23\rangle\langle 34\rangle\langle 2|3+4|5]\langle 1|P_V|5]}\,. \tag{4.18}$$

Using the parity and line reversal symmetries of eqs. (4.6) and (4.14) we can easily obtain expressions for the other 12 helicity configurations.

All the amplitudes presented in this section have been numerically checked against Feynman-diagram based calculations.

4.2 n-point Currents

It is also possible to construct single vector boson currents with n partons in the helicity configurations with maximal helicity violation, next-to-maximal helicity violation and beyond. The current for vector boson decaying to a quark pair and any number of positive helicity gluons has been known for some time [30],

$$S_\mu(1_q^-,2^+,\ldots,(n-1)^+,n_{\bar q}^+) = (-1)^n\,\frac{\langle 1|\sigma_\mu P_V|1\rangle}{\prod_{\alpha=1}^{n-1}\langle \alpha\,\alpha+1\rangle}\,. \tag{4.19}$$

As usual P_V is the momentum of the vector boson and $P_V = -p_1-\ldots-p_n$. This can be easily proved by induction using on-shell recursion relations. The fact that any pure QCD amplitude with less than two negative helicities is zero guarantees that the only contribution to the n-point current involves an $(n-1)$-point current and an on-shell (complex) 3-gluon vertex, as shown in figure 2. This is the first non-vanishing helicity amplitude and hence it is the MHV current. For completeness, the other MHV-type currents are given by,
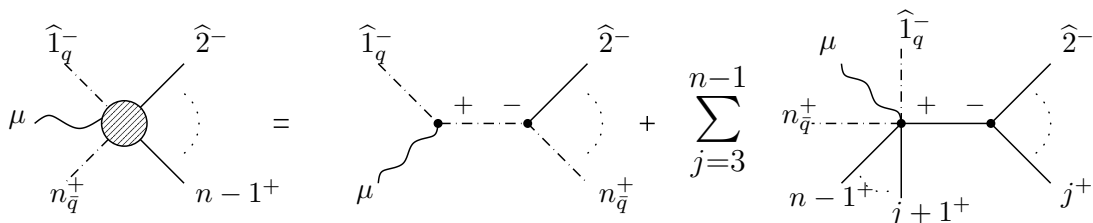
Figure 3: Decomposition of the n-point vector boson current with $n-3$ positive helicity gluons and two adjacent negative helicities.

$$S_\mu(1_q^+, 2^+, \ldots, (n-1)^+, n_{\bar{q}}^-) = -(-1)^n \frac{\langle n|\sigma_\mu P_V|n\rangle}{\prod_{\alpha=1}^{n-1}\langle \alpha\,\alpha+1\rangle},$$

$$S_\mu(1_q^+, 2^-, \ldots, (n-1)^-, n_{\bar{q}}^-) = -\frac{[1|P_V\bar{\sigma}_\mu|1]}{\prod_{\alpha=1}^{n-1}[\alpha\,\alpha+1]},$$

$$S_\mu(1_q^-, 2^-, \ldots, (n-1)^-, n_{\bar{q}}^+) = \frac{[n|P_V\bar{\sigma}_\mu|n]}{\prod_{\alpha=1}^{n-1}[\alpha\,\alpha+1]}. \tag{4.20}$$

It is interesting to note that eq. (4.19) allows us to immediately write down compact expressions for the NMHV currents with both adjacent and non-adjacent minuses. If we mark the two negative helicity particles then each sub-amplitude in the recursion relation will contain at most 2 negative helicities. Figure 3 shows the decomposition into a sum of sub-diagrams. We draw a pure QCD amplitude on the right of each diagram which must contain at least two negative helicity particles, this fixes the helicity on the right of each propagator to be negative. Helicity conservation then ensures that the vector currents have only one negative helicity, the marked quark, and so are MHV currents. We can therefore use (4.19) to write down the NMHV current, marking $i = 1$ and $j = 2$:

$$S_\mu(1_q^-, 2^-, 3^+, \ldots, (n-1)^+, n_{\bar{q}}^+) = \frac{-(-1)^n}{\prod_{\alpha=2}^{n-1}\langle \alpha\,\alpha+1\rangle}\left(\frac{\langle 2|K_{3,n}P_V\sigma_\mu K_{3,n}|2\rangle\langle 2n\rangle}{s_{2,n}\langle n|K_{2,n-1}|1]}\right.$$

$$\left. + \sum_{j=3}^{n-1}\frac{\langle 2|K_{3,j}K_{1,j}P_V\bar{\sigma}_\mu K_{1,j}K_{3,j}|2\rangle\langle 2|K_{3,j}|1]\langle j\,j+1\rangle}{s_{2,j}s_{1,j}\langle j|K_{2,j-1}|1]\langle j+1|K_{2,j}|1]}\right), \tag{4.21}$$

This expression is the n-parton generalisation of equations (4.12) and (4.16).

We can also consider the case where the helicity along the quark line is flipped. This is a special case as we can still eliminate all contributions from NMHV vertices. The result is,

$$S_\mu(1_q^-, 2^+, \ldots, (n-2)^+, (n-1)^-, n_{\bar{q}}^+) = \frac{(-1)^n}{\prod_{\alpha=1}^{n-2}\langle \alpha\,\alpha+1\rangle}\left(\frac{\langle 1n-1\rangle^3[n|\bar{\sigma}_\mu P_V|n]}{s_{1,n-1}\langle 1|P_V|n]}\right.$$

$$\left. + \sum_{j=1}^{n-3}\frac{\langle n-1|K_{j+1,n-2}|n]^3\langle 1|\sigma_\mu P_V|1]\langle j\,j+1\rangle}{s_{j+1,n}s_{j+1,n-1}\langle j+1|K_{j+1,n-1}|n]\langle j|K_{j+1,n-1}|n]}\right), \tag{4.22}$$

matching equations (4.13) and (4.18) when $n = 4$ and 5 respectively.

For NMHV currents with non-adjacent negative helicities we can re-use the above result to find the amplitude where the negative helicities are separated by one positive helicity.
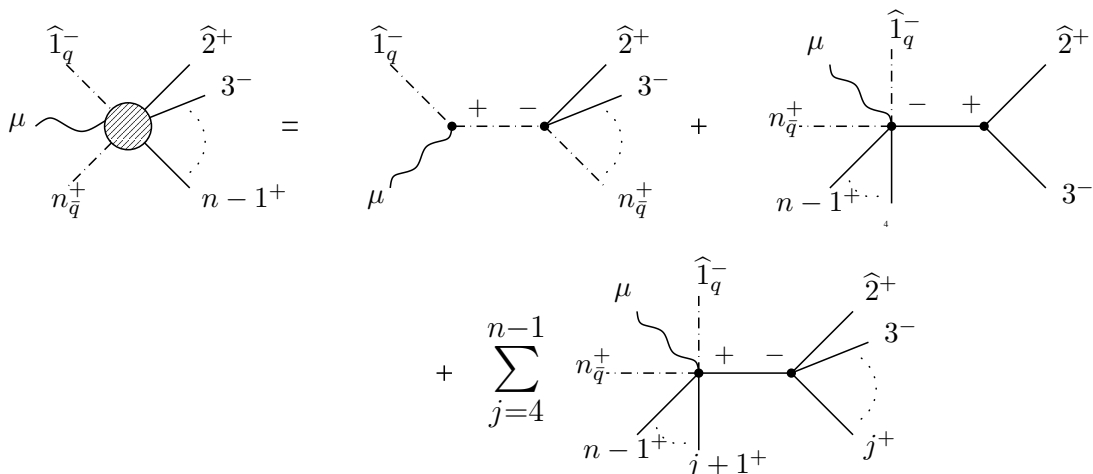
Figure 4: Decomposition of the n-point vector boson current with $n-3$ positive helicity gluons and two negative helicities, separated by a single gluon with positive helicity. The second contribution involves the NMHV current with adjacent negative helicities given in (4.21).

The corresponding diagrams are shown in figure 4 where we mark $i = 2$ and $j = 1$,

$$
S_\mu(1_q^-, 2^+, 3^-, 4^+, \ldots, n-1^+, n_{\bar{q}}^+) = \frac{\langle 1|P_V \bar{\sigma}_\mu|1\rangle \langle 3|K_{4,n}|2]^3 \langle 3n\rangle}{s_{2,n} s_{3,n} \langle 1|K_{2,n}K_{4,n}|3\rangle \langle n|K_{3,n-1}|2] \prod_{\alpha=3}^{n-1} \langle \alpha\,\alpha+1\rangle}
$$
$$
+ \sum_{j=4}^{n-1} \frac{\langle 1|\sigma_\mu P_V|1\rangle \langle 3|K_{4,j}|2]^4 \langle j\,j+1\rangle}{s_{2,j} s_{3,j} \langle 1|K_{2,j}K_{4,j}|3\rangle \langle j|K_{3,j-1}|2] \langle j+1|K_{3,j}|2] \prod_{\alpha=3}^{n-1} \langle \alpha\,\alpha+1\rangle}
$$
$$
+ S_\mu(\widehat{1}_q^-, \widehat{P}^-, 4^+, \ldots, n-1^+, n_{\bar{q}}^+) \frac{\langle 31\rangle^3}{\langle 12\rangle \langle 23\rangle \bar{w}^2} \tag{4.23}
$$

Substituting eq. (4.21) and simplifying the shifts results in the following expression:

$$
S_\mu(1_q^-, 2^+, 3^-, 4^+, \ldots, n-1^+, n_{\bar{q}}^+) = \frac{1}{\prod_{\alpha=3}^{n-1} \langle \alpha\,\alpha+1\rangle} \left(\frac{\langle 1|P_V \bar{\sigma}_\mu|1\rangle \langle 3|K_{4,n}|2]^3 \langle 3n\rangle}{s_{2,n} s_{3,n} \langle 1|K_{2,n}K_{4,n}|3\rangle \langle n|K_{3,n-1}|2]} \right.
$$
$$
+ \sum_{j=4}^{n-1} \frac{\langle 1|\sigma_\mu P_V|1\rangle \langle 3|K_{4,j}|2]^4 \langle j\,j+1\rangle}{s_{2,j} s_{3,j} \langle 1|K_{2,j}K_{4,j}|3\rangle \langle j|K_{3,j-1}|2] \langle j+1|K_{3,j}|2]}
$$
$$
+ \frac{\langle 13\rangle^3 \langle n3\rangle \langle 3|K_{4,n}P_V \sigma_\mu K_{4,n}|3\rangle}{\langle 1|K_{2,n}K_{4,n}|3\rangle \langle n|K_{1,n-1}K_{12}|3\rangle \langle 12\rangle \langle 23\rangle}
$$
$$
\left. + \sum_{j=4}^{n-1} \frac{\langle 13\rangle^3 \langle 3|K_{4,j}K_{1,j}P_V \bar{\sigma}_\mu K_{1,j}K_{4,j}|3\rangle \langle 3|K_{12}K_{4,j}|3\rangle \langle j\,j+1\rangle}{s_{1,j} \langle 1|K_{2,j}K_{4,j}|3\rangle \langle j|K_{1,j-1}K_{12}|3\rangle \langle j+1|K_{1,j}K_{12}|3\rangle \langle 12\rangle \langle 23\rangle} \right). \tag{4.24}
$$

Remaining NMHV currents can be constructed in a similar way. We can keep adding an extra positive helicity separating the negative helicities. We have checked that the n-parton result in eq. (4.24) agrees with eqs. (4.13) and (4.17) for $n = 4$ and $n = 5$ respectively.

As a final example we consider the NNMHV current with three adjacent negative helicities.
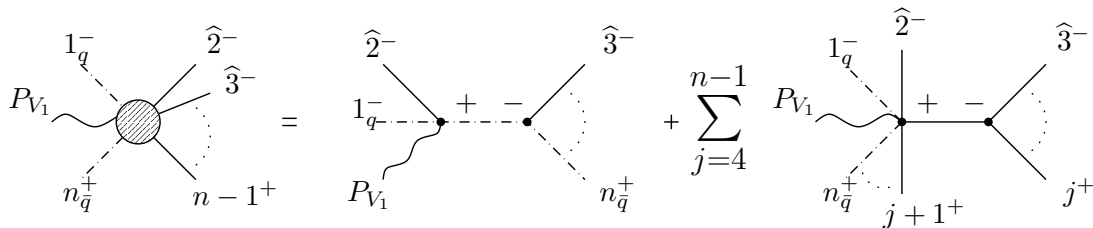
Figure 5: Decomposition of the single vector boson current with three consecutive negative helicites. The recursion re-uses the current with two consecutive negative helicites.
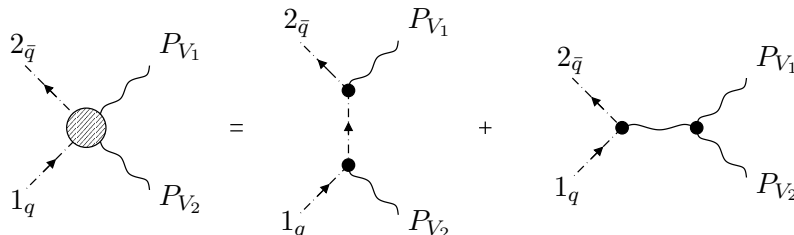

Figure 6: The two Feynman diagrams contributing to the amplitude with two massive vector bosons and a quark pair.

By marking particles $i = 2$ and $j = 3$ we ensure that only the NMHV current (4.21) is needed. Explicit calculation yields,

$$S_\mu(1_q^-, 2^-, 3^-, 4^+, \ldots, n-1^+, n_{\bar{q}}^+) = \frac{(-1)^n}{\prod_{\alpha=3}^{n-1}\langle \alpha\, \alpha+1\rangle}\left(\frac{\langle 3n\rangle\langle 3|K_{4,n}P_V\sigma_\mu K_{4,n}|3\rangle}{s_{3,n}[12]\langle n|K_{3,n-1}|2]}\right.$$

$$+ \sum_{j=4}^{n-1}\frac{\langle 3|K_{4,j}|1]\langle 3|K_{4,j}K_{1,j}P_V\sigma_\mu K_{1,j}K_{4,j}|3\rangle\langle j\, j+1\rangle}{s_{3,j}s_{1,j}[12]\langle j+1|K_{2,j}|1]\langle j|K_{3,j-1}|2]}$$

$$- \sum_{j=4}^{n-1}\sum_{l=j+1}^{n-1}\frac{\langle 3|K_{4,j}K_{2,j}K_{2,l}|1]\langle 3|K_{4,j}K_{2,j}K_{2,l}K_{1,l}P_V\bar{\sigma}_\mu K_{1,l}K_{2,l}K_{2,j}K_{4,j}|3\rangle\langle j\, j+1\rangle\langle l\, l+1\rangle}{s_{1,l}s_{2,l}s_{2,j}s_{3,j}\langle l|K_{2,l-1}|1]\langle l+1|K_{2,l}|1]\langle j+1|K_{3,j}|2]\langle j|K_{3,j-1}|2]}$$

$$\left.- \sum_{j=4}^{n-1}\frac{\langle 3|K_{4,j}K_{2,j}|n\rangle\langle 3|K_{4,j}K_{2,j}K_{2,n}P_V\bar{\sigma}_\mu K_{2,n}K_{2,j}K_{4,j}|3\rangle\langle j\, j+1\rangle}{s_{3,j}s_{2,n}s_{2,j}\langle n|K_{2,n-1}|1]\langle j+1|K_{3,j}|2]\langle j|K_{3,j-1}|2]}\right). \quad (4.25)$$

We have explicitly checked eq. (4.25) for up to six partons.

By repeated use of the recursion formulae, further n-point currents may be obtained.

5. Double vector boson currents

We now turn to double vector boson currents $S_{\mu\nu}$. We start by considering the smallest amplitude of this type, the one with only two partons, $S_{\mu\nu}(q, \bar{q})$. One might expect that on-shell recursion relations can be used to derive $S_{\mu\nu}(q, \bar{q})$ from two single vector boson amplitudes $S_\mu(q, \bar{q})$. However there is a difficulty in writing down such a recursion relation. We cannot mark the two massless particles in $S_{\mu\nu}(q, \bar{q})$ since it is known that

marking adjacent massless quarks results in a non-vanishing boundary contribution [21] to the amplitude. Choosing to mark massive particles also leads to (unnecessary) technical complications. It is actually much simpler to derive the four-point amplitude $S_{\mu\nu}(q, \bar{q})$ from Feynman diagrams and use this four-point amplitude as a new primitive vertex in further recursive calculations of $S_{\mu\nu}(1_q, \ldots, n_{\bar{q}})$.

In fact, we will use a more elegant approach. In general, there are two Feynman diagrams contributing to $S_{\mu\nu}(q, \bar{q})$, as shown in figure 6. We could evaluate both diagrams and use the whole amplitude as a building block for larger amplitudes, however, it is much more efficient to split the calculation into two parts, as considered in [52], in order to re-use the single vector boson currents computed in section 4.

The first part corresponds to the second diagram in figure 6, it contains the non-Abelian three-vertex of massive vector bosons. We can compute such contributions to a generic $S_{\mu\nu}(1_q, \ldots, n_{\bar{q}})$ by contracting a single vector boson current $S_{\mu}(1_q, \ldots, n'_{\bar{q}})$ of the previous section with the colour ordered Feynman three-point vertex. This approach was used to calculate the non-Abelian contribution to $S_{\mu\nu}(q, g, \bar{q})$ in reference [15]. Note that if one is dealing with uncharged gauge bosons which have no self-coupling, for example Z bosons, this term is trivially zero.

The second part does not contain a non-Abelian three-vertex of vector bosons, it corresponds to the first diagram in figure 6. This second Abelian contribution to a generic $S_{\mu\nu}(1_q, \ldots, n_{\bar{q}})$ can then be evaluated using on-shell recursion relations.

Thus, guided by figure 6 we represent the colour ordered double current with n partons in the form [52]:

$$S_{\mu\nu}(1_q, \ldots, n_{\bar{q}}) = T^{(3)}_{\mu\nu\rho}(P_{V_1}, P_{V_2}, -P) \frac{1}{(P^2 - M_P^2)} S^{\rho}(1_q, \ldots, n_{\bar{q}})$$
$$+ S^{Abelian}_{\mu\nu}(1_q, \ldots, n_{\bar{q}}). \tag{5.1}$$

Here

$$T^{(3)}_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) = g_{\mu_1\mu_2}(p_1 - p_2)_{\mu_3} + g_{\mu_2\mu_3}(p_2 - p_3)_{\mu_1} + g_{\mu_3\mu_1}(p_3 - p_1)_{\mu_2} \tag{5.2}$$

is the colour-ordered three-vertex of massive vector bosons, with all momenta defined to be in-going.

The *primitive vertex* for the Abelian contribution is given by the first Feynman diagram in figure 6 which evaluates to:

$$S^{Abelian}_{\mu\nu}(1_q^-, 2_{\bar{q}}^+) = \frac{1}{s_{1P_{V_2}}} \langle 1|\sigma_\nu(1 + P_{V_2})\sigma_\mu|2]. \tag{5.3}$$

The remaining non-Abelian part of this four-point amplitude is determined by the first line of (5.1)

$$S^{non-Abelian}_{\mu\nu}(1_q, 2_{\bar{q}}) = T^{(3)}_{\mu\nu\rho}(P_{V_1}, P_{V_2}, -P) \frac{1}{(P^2 - M_P^2)} S^{\rho}(1_q, 2_{\bar{q}}) \tag{5.4}$$

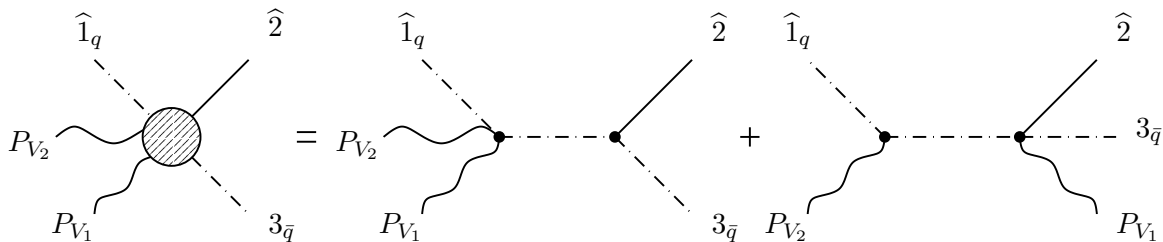where $S^{\rho}(1_q, 2_{\bar{q}})$ is given in (4.5).

Figure 7: Recursive decomposition of the five-point Abelian amplitude for two vector bosons, a quark pair and a gluon using the recursion relation.

In general one needs to determine only the Abelian components of the n-parton double currents $S_{\mu\nu}(1_q, \ldots, n_{\bar{q}})$, the non-Abelian components are fully determined by the first line of (5.1) in terms of the known single currents.

Abelian components are characterised by having massive vector bosons only on external lines, and they can always be calculated recursively. In general, in order to calculate the Abelian part of any double current, $S_{\mu\nu}^{Abelian}(1_q, 2, \ldots, n_{\bar{q}}^+)$, one needs to draw all recursive decompositions of this current such that the internal line is a quark or a gluon and not a massive vector boson.

First we calculate the two five-point amplitudes by marking the quark and adjacent gluon. The recursion relation for $S_{\mu\nu}^{Abelian}(1_q, 2, 3_{\bar{q}})$ is depicted in figure 7. It yields,

$$
\begin{aligned}
S_{\mu\nu}^{Abelian}(1_q^-, 2^-, 3_{\bar{q}}^+) = &+ \frac{[13][3|\bar{\sigma}_\mu(1+P_{V_2})\bar{\sigma}_\nu(1+2)|3]}{[12][23][3|P_{V_1}P_{V_2}|1]} \\
&+ \frac{\langle 2|\sigma_\mu|3][1|\bar{\sigma}_\nu(1+2)|3]}{[12][3|P_{V_1}P_{V_2}|1]} \\
&- \frac{\langle 2|(1+P_{V_2})\bar{\sigma}_\nu P_{V_2}(1+P_{V_2})|2\rangle[3|P_{V_1}\sigma_\mu|3]}{s_{1P_{V_2}} s_{3P_{V_1}}[3|P_{V_1}P_{V_2}|1]} ,
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
S_{\mu\nu}^{Abelian}(1_q^-, 2^+, 3_{\bar{q}}^+) = &\frac{\langle 13\rangle\langle 1|(2+3)\bar{\sigma}_\mu(1+P_{V_2})\bar{\sigma}_\nu|1\rangle}{\langle 12\rangle\langle 23\rangle\langle 1|P_{V_2}P_{V_1}|3\rangle} \\
&+ \frac{\langle 1|\sigma_\nu|2]\langle 1|(2+3)\bar{\sigma}_\mu|1\rangle}{\langle 12\rangle\langle 1|P_{V_2}P_{V_1}|3\rangle} \\
&- \frac{\langle 1|P_{V_2}\bar{\sigma}_\nu|1\rangle[2|(1+P_{V_2})P_{V_1}\bar{\sigma}_\mu(1+P_{V_2})|2]}{s_{1P_{V_2}} s_{3P_{V_1}}\langle 1|P_{V_2}P_{V_1}|3\rangle} .
\end{aligned}
\tag{5.6}
$$

These formulae agree with the results of [52] after appropriate permutations of the labels and correcting for propagator conventions. The $S_{\mu\nu}(+; \pm; -)$ configurations can be obtained from eqs. (4.11) by using either parity (4.7) or the "line reversal" symmetry,

$$
S_{\mu\nu}(1_q^\lambda, 2^{h_2}, \ldots, n-1^{h_{n-1}}, n_{\bar{q}}^{-\lambda}) = (-1)^{n+1} S_{\nu\mu}(n_q^{-\lambda}, n-1^{h_{n-1}}, \ldots, 2^{h_2}, 1_{\bar{q}}^\lambda) .
\tag{5.7}
$$

Finally we give results for the six point amplitudes $S_{\mu\nu}^{Abelian}(1_q, 2, 3, 4_{\bar{q}})$. Taking the generalised parity relation (4.7) and the line-reversal identity (5.7), there are three independent helicity configurations. Again we use on-shell recursion relations and mark the

quark and adjacent gluon. Choosing $i = 2$ and $j = 1$ we find,

$$S_{\mu\nu}^{Abelian}(1_q^-, 2^+, 3^+, 4_{\bar{q}}^+) = -\frac{\langle 14\rangle\langle 1|\sigma_\nu|2]\langle 1|(2+3+4)\bar{\sigma}_\mu|1\rangle}{\langle 12\rangle\langle 13\rangle\langle 34\rangle\langle 1|P_{V_2}P_{V_1}|4]}$$

$$- \frac{\langle 14\rangle\langle 1|(2+3+4)\bar{\sigma}_\mu(1+P_{V_2})\bar{\sigma}_\nu|1\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 1|P_{V_2}P_{V_1}|4]}$$

$$+ \frac{\langle 1|(2+3)\bar{\sigma}_\nu|1\rangle\langle 1|(P_{V_1}+P_{V_2})\bar{\sigma}_\mu|1\rangle}{\langle 12\rangle\langle 13\rangle\langle 23\rangle\langle 1|P_{V_2}P_{V_1}|4]}$$

$$+ \frac{\langle 1|P_{V_2}\bar{\sigma}_\nu|1\rangle\langle 1|(2+3)(4+P_{V_1})P_{V_1}\bar{\sigma}_\mu(4+P_{V_1})(2+3)|1\rangle}{s_{4P_{V_1}}\langle 12\rangle\langle 23\rangle\langle 1|P_{V_2}P_{V_1}|4]\langle 1|P_{V_2}(4+P_{V_1})|3]}$$

$$+ \frac{\langle 1|P_{V_2}\bar{\sigma}_\nu|1\rangle[2|(1+P_{V_2})P_{V_1}\bar{\sigma}_\mu(1+P_{V_2})|2]}{s_{1P_{V_2}}s_{12P_{V_2}}\langle 34\rangle\langle 1|P_{V_2}(4+P_{V_1})|3]}, \tag{5.8}$$

and

$$S_{\mu\nu}^{Abelian}(1_q^-, 2^+, 3^-, 4_{\bar{q}}^+) = -\frac{\langle 13\rangle^3\langle 3|1+2|4][4|\bar{\sigma}_\mu(1+P_{V_2})\bar{\sigma}_\nu(1+2+3)|4]}{s_{123}\langle 12\rangle\langle 23\rangle\langle 1|2+3|4]\langle 3|(1+2)P_{V_2}P_{V_1}|4]}$$

$$- \frac{\langle 13\rangle^3\langle 3|(4+P_{V_1})\bar{\sigma}_\nu P_{V_2}(4+P_{V_1})|3][4|P_{V_1}\sigma_\mu|4]}{s_{4P_{V_1}}\langle 12\rangle\langle 23\rangle\langle 1|P_{V_2}(4+P_{V_1})|3]\langle 3|(1+2)P_{V_2}P_{V_1}|4]}$$

$$- \frac{\langle 13\rangle^2\langle 1|\sigma_\mu|4]\langle 3|1+2|4][2|\bar{\sigma}_\nu(1+2+3)|4]}{s_{123}\langle 12\rangle\langle 1|2+3|4]\langle 3|(1+2)P_{V_2}P_{V_1}|4]}$$

$$+ \frac{\langle 13\rangle^2\langle 3|\sigma_\mu|4]\langle 3|(1+2)\bar{\sigma}_\nu(1+2+3)|4]}{s_{123}\langle 12\rangle\langle 23\rangle\langle 3|(1+2)P_{V_2}P_{V_1}|4]}$$

$$- \frac{[24]^3\langle 1|3+4|2]\langle 1|(2+3+4)\bar{\sigma}_\mu(1+P_{V_2})\bar{\sigma}_\nu|1\rangle}{s_{234}\langle 1|P_{V_2}P_{V_1}(3+4)|2]\langle 1|2+3|4][23][34]}$$

$$- \frac{[24]^3\langle 1|\sigma_\nu|2]\langle 1|(2+3+4)\bar{\sigma}_\mu|1\rangle}{\langle 1|P_{V_2}P_{V_1}(3+4)|2]\langle 1|2+3|4][23][34]}$$

$$- \frac{[24]^3\langle 1|P_{V_2}\bar{\sigma}_\nu|1\rangle[2|(1+P_{V_2})P_{V_1}\bar{\sigma}_\mu(1+P_{V_2})|2]}{s_{1P_{V_2}}\langle 1|P_{V_2}P_{V_1}(3+4)|2][23][34][2|(1+P_{V_2})P_{V_1}|4]}$$

$$- \frac{\langle 3|1+P_{V_2}|2]^3\langle 1|P_{V_2}\bar{\sigma}_\nu|1\rangle[4|P_{V_1}\sigma_\mu|4]}{s_{4P_{V_1}}s_{1P_{V_2}}s_{12P_{V_2}}\langle 1|P_{V_2}(4+P_{V_1})|3][2|(1+P_{V_2})P_{V_1}|4]}. \tag{5.9}$$

For the last amplitude we choose $i = 1$ and $j = 2$ which yields the following expression,

$$S_{\mu\nu}^{Abelian}(1_q^-, 2^-, 3^+, 4_{\bar{q}}^+) = -\frac{\langle 24\rangle[1|\bar{\sigma}_\nu(P_{V_1}+P_{V_2})(3+4)|2]\langle 2|(3+4)\bar{\sigma}_\mu|2]}{\langle 23\rangle\langle 34\rangle\langle 2|(3+4)P_{V_1}P_{V_2}|1]\langle 4|2+3|1]}$$

$$- \frac{\langle 24\rangle\langle 2|3+4|1]\langle 2|(3+4)\bar{\sigma}_\mu(1+P_{V_2})\bar{\sigma}_\nu(P_{V_1}+P_{V_2})(3+4)|2]}{s_{234}\langle 23\rangle\langle 34\rangle\langle 2|(3+4)P_{V_1}P_{V_2}|1]\langle 4|2+3|1]}$$

$$+ \frac{\langle 24\rangle\langle 2|(1+P_{V_2})\bar{\sigma}_\nu P_{V_2}(1+P_{V_2})|2]\langle 2|(3+4)P_{V_1}\sigma_\mu(3+4)|2]}{s_{1P_{V_2}}\langle 23\rangle\langle 34\rangle\langle 2|(1+P_{V_2})P_{V_1}|4]\langle 2|(3+4)P_{V_1}P_{V_2}|1]}$$

$$- \frac{[13][3|1+2|4][3|(1+2)(P_{V_1}+P_{V_2})\bar{\sigma}_\mu(1+P_{V_2})\bar{\sigma}_\nu(1+2)|3]}{s_{123}\langle 4|2+3|1][12][23][3|(1+2)P_{V_2}P_{V_1}|4]}$$

$$- \frac{[13][3|(1+2)P_{V_2}\bar{\sigma}_\nu(1+2)|3][3|(4+P_{V_1})P_{V_1}\bar{\sigma}_\mu(4+P_{V_1})|3]}{s_{4P_{V_1}}[12][23][3|(4+P_{V_1})P_{V_2}|1][3|(1+2)P_{V_2}P_{V_1}|4]}$$

$$
\begin{aligned}
&- \frac{\langle 2|1+P_{V_2}|3]\langle 2|(1+P_{V_2})\bar{\sigma}_\nu P_{V_2}(1+P_{V_2})|2\rangle[3|(4+P_{V_1})P_{V_1}\bar{\sigma}_\mu(4+P_{V_1})|3]}{s_{4P_{V_1}}s_{1P_{V_2}}s_{12P_{V_2}}\langle 2|(1+P_{V_2})P_{V_1}|4\rangle[1|P_{V_2}(4+P_{V_1})|3]} \\
&- \frac{[3|(1+2)(P_{V_1}+P_{V_2})\bar{\sigma}_\mu|2\rangle[3|1+2|4\rangle[1|\bar{\sigma}_\nu(1+2)|3]}{s_{123}\langle 4|2+3|1][12][3|(1+2)P_{V_2}P_{V_1}|4\rangle} \\
&+ \frac{[3|\bar{\sigma}_\nu(1+2)|3][3|(1+2)(P_{V_1}+P_{V_2})\bar{\sigma}_\mu(1+2)|3]}{s_{123}[12][23][3|(1+2)P_{V_2}P_{V_1}|4\rangle}.
\end{aligned}
\tag{5.10}
$$

The procedure described here can straightforwardly be generalised to processes involving three or more vector bosons. In each case, there will be a mixture of terms that either involve a triple or quartic gauge boson vertex (non-Abelian) or a new multi-gauge boson current (Abelian). The non-abelian contribution is straightfoward and involves currents with coupling that couples currents involving fewer gauge bosons. These are in principle known. For each additional vector boson, the Abelian contribution must be recomputed. There will be a new primitive vertex which can be obtained directly from the single (colour-ordered) Feynman diagram.

This is illustrated in figure 8 for three vector bosons. The first diagram yields a new primitive vertex $S_{\mu_1\mu_2\mu_3}(1_q, 2_{\bar{q}})$ which forms the seed for recursively calculating the Abelian contribution to the amplitude. The other three (Non-Abelian) graphs can be straightforwardly obtained by reusing the single and double vector boson currents. The colour ordered triple current with n partons is thus,

$$
\begin{aligned}
S_{\mu_1\mu_2\mu_3}(1_q,\ldots,n_{\bar{q}}) = {}& S^{Abelian}_{\mu_1\mu_2\mu_3}(1_q,\ldots,n_{\bar{q}}) \\
&+ T^{(3)}_{\mu_1\mu_2\rho}(P_{V_1},P_{V_2},-P_{12})\frac{1}{(P_{12}^2-M_{P_{12}}^2)}S^{\rho\mu_3}(1_q,\ldots,n_{\bar{q}}) \\
&+ T^{(3)}_{\mu_2\mu_3\rho}(P_{V_2},P_{V_3},-P_{23})\frac{1}{(P_{23}^2-M_{P_{23}}^2)}S^{\mu_1\rho}(1_q,\ldots,n_{\bar{q}}) \\
&+ T^{(4)}_{\mu_1\mu_2\mu_3\rho}(P_{V_1},P_{V_2},P_{V_3},-P_{123})\frac{1}{(P_{123}^2-M_{P_{123}}^2)}S^{\rho}(1_q,\ldots,n_{\bar{q}}). \quad (5.11)
\end{aligned}
$$

The colour ordered quartic gauge boson vertex is given by,

$$
T^{(4)}_{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) = 2g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_2}g_{\mu_3\mu_4}.
\tag{5.12}
$$

6. Recursion relations for massive particles with spin on internal lines

So far we have been considering application of recursion relations where massive particles with spin were absent from the internal lines. In other words, we have been able to set up the recursive calculations of double vector boson currents in such a way that the massive vector bosons were playing the role of 'external sources' in the left and in the right hand vertices, but were not propagating through the recursive diagram. We now would like to show how to use recursion relations also for propagating massive particles. In our earlier work [28] we have accomplished this for massive scalars, and now we want to generalise this approach to massive particles with spin.
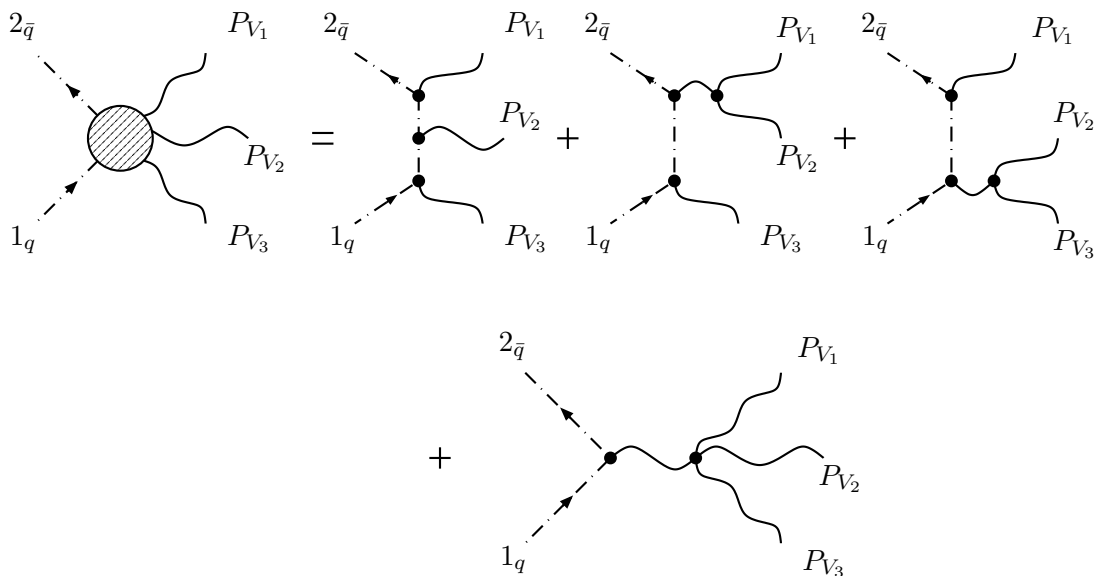
Figure 8: Contributions to the amplitude with three massive vector bosons and a quark anti-quark pair.

The main difference between internal massive scalars of ref. [28] and internal massive fermions or vector bosons is that the latter have more than one polarisation or spin state. In the standard recursion relation (2.1) all particles are assumed to be in a state with fixed helicity, and there is a summation over all these states. We want to avoid using helicity states for internal massive particles and instead to use a more natural basis of states

In this section we will describe a new way to implement the recursion relation in this case and will illustrate its use by calculating a simple amplitude of two heavy quarks scattering into two gluons.

The main point here is that the sum over helicities h of the internal particle in the standard recursion relation,

$$\mathcal{A}_n(p_1, \ldots, p_n) = \sum_{\text{partitions}} \sum_h \mathcal{A}_L(p_r, \ldots, \widehat{p}_i, \ldots, p_s, -\widehat{P}^h) \frac{1}{P^2 - m_P^2}$$
$$\times \, \mathcal{A}_R(\widehat{P}^{-h}, p_{s+1}, \ldots, \widehat{p}_j, \ldots, p_{r-1}) \,, \qquad (6.1)$$

can be replaced by the sum over all of the spin states, rather than helicity quantum numbers which are not well suited for massive particles. So, we first replace the sum over helicities by the sum over appropriately defined spin states. For massive fermions this is the conventional spin sum:

$$\sum_{s=1,2} u_s(p)\bar{u}_s(p) = \not{p} + m_p \qquad (6.2)$$

$$\sum_{s=1,2} v_s(p)\bar{v}_s(p) = \not{p} - m_p \qquad (6.3)$$
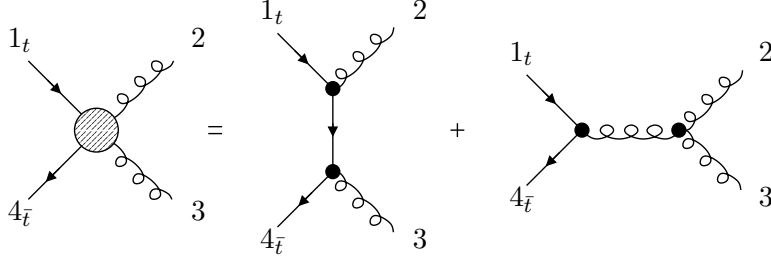
– 16 –

Figure 9: The two Feynman diagrams contributing to the amplitude $\mathcal{A}_4(1_t, 2, 3, 4_{\bar{t}})$.

The remaining spinors and polarisation vectors of the external massive particles can be left unfixed and simplified after squaring the amplitude with the spin sums in the conventional way.

Using this in the recursion relation in which a massive quark propagates between the two diagrams we have

$$\sum_s \mathcal{A}_L(p_{r;\bar{q}}, \ldots, \widehat{p}_i, \ldots, p_s, -\widehat{P}_q^s) \frac{1}{P^2 - m_P^2} \mathcal{A}_R(\widehat{P}_{\bar{q}}^{-s}, p_{s+1}, \ldots, \widehat{p}_j, \ldots, p_{r-1;q})$$

$$= \mathcal{A}_L(p_{r;\bar{q}}, \ldots, \widehat{p}_i, \ldots, p_s, -\widehat{P}^*) \frac{\widehat{\slashed{P}} + m_P}{P^2 - m_P^2} \mathcal{A}_R(\widehat{P}^*, p_{s+1}, \ldots, \widehat{p}_j, \ldots, p_{r-1;q}) \tag{6.4}$$

where P^* indicates the external spinor wave-function has been stripped off this amplitude. In this way we can use the benefits of using the recursion relations to provide reasonably compact formulae for amplitudes with massive particles.

6.1 Example: calculation of $\mathcal{A}_4(1_t, 2, 3, 4_{\bar{t}})$

We now compute the four point amplitude of a top quark pair scattering to two gluons as an example of the method described above. We use on-shell recursion relations and mark two massless gluons. This leads to a single recursive diagram with a massive fermion propagator. We will show that the contribution of this single diagram precisely matches the two Feynman diagrams for this process shown in figure 9. With all particles outgoing the recursion relation result is,

$$\mathcal{A}(1_t, 2, 3, 4_{\bar{t}}) = \frac{1}{(P^2 - m_t^2)} \bar{u}(p_1) \slashed{\epsilon}(p_2, \xi_2) \left(\sum_s u_s(\widehat{P}) \bar{u}_s(\widehat{P}) \right) \slashed{\epsilon}(p_3, \xi_3) v(p_4). \tag{6.5}$$

Here $P = p_1 + p_2$ is the momentum on the internal line and ξ_2, ξ_3 are reference spinors necessary to specify gluon polarisation vectors $\epsilon^{\pm}$. We will use the Weyl representation of the Dirac γ-matrices and polarisation vectors,

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}, \quad \slashed{\epsilon}^+(p, \xi) = \frac{1}{\langle \xi p \rangle} \begin{pmatrix} 0 & |\xi\rangle[p| \\ |p]\langle\xi| & 0 \end{pmatrix}, \quad \slashed{\epsilon}^-(p, \xi) = \frac{1}{[\xi p]} \begin{pmatrix} 0 & |p\rangle[\xi| \\ |\xi]\langle p| & 0 \end{pmatrix}. \tag{6.6}$$

First we consider the case where the gluons have opposite helicity, $\mathcal{A}(1_t, 2^-, 3^+, 4_{\bar{t}})$. It is convenient to choose $\xi_2 = p_3$ and $\xi_3 = p_2$ so that,

$$\mathcal{A}(1_t, 2^-, 3^+, 4_{\bar{t}}) = \frac{1}{(P^2 - m_t^2)s_{23}} \, \bar{u}(p_1) \begin{pmatrix} 0 & |\hat{2}\rangle[3| \\ |3]\langle\hat{2}| & 0 \end{pmatrix} \begin{pmatrix} m_t & \hat{\slashed{P}} \\ \hat{\slashed{P}} & m_t \end{pmatrix} \begin{pmatrix} 0 & |2\rangle[\hat{3}| \\ |\hat{3}]\langle 2| & 0 \end{pmatrix} v(p_4). \tag{6.7}$$

We choose the marking prescription $i = 3$ and $j = 2$ and this ensures that the shifts on the polarisation vectors disappear. It can also be seen that the shift in $\hat{P}$ is also killed by either of the two polarisation vectors and hence we can erase all the hats in equation (6.7). This is then exactly equivalent to the first diagram of figure 9. It can be easily shown using $\epsilon^-(2,3) \cdot \epsilon^+(3,2) = 0$ that, with this particular choice of reference momenta, that the remaining second Feynman diagram gives a vanishing contribution and so our recursion relation result is in agreement with the Feynman diagrams answer.

The amplitude with both gluons of negative helicity $\mathcal{A}(1_t, 2^+, 3^+, 4_{\bar{t}})$ is of a non-MHV type and it provides another interesting test of the recursion relation, which this time requires a little algebra. The recursion relation reads:

$$\mathcal{A}(1_t, 2^+, 3^+, 4_{\bar{t}}) = \frac{1}{(P^2 - m_t^2)\langle 23\rangle^2} \bar{u}(p_1) \begin{pmatrix} 0 & |3\rangle[\hat{2}| \\ |\hat{2}]\langle 3| & 0 \end{pmatrix} \begin{pmatrix} m_t & \hat{\slashed{P}} \\ \hat{\slashed{P}} & m_t \end{pmatrix} \begin{pmatrix} 0 & |2\rangle[\hat{3}| \\ |\hat{3}]\langle 2| & 0 \end{pmatrix} v(p_4). \tag{6.8}$$

Again, choosing $i = 3$ and $j = 2$ removes the shifts on the propagator and the polarisation vector of gluon p_3. However in this case all the shifts do not vanish as $|\hat{2}] = |2] - z|3]$ hence we are left with the exact expression for the first Feynman diagram *plus* an extra term coming from the surviving shifts:

$$-\frac{z}{(P^2 - m_t^2)\langle 23\rangle^2} \bar{u}(p_1) \begin{pmatrix} 0 & |3\rangle[3| \\ |3]\langle 3| & 0 \end{pmatrix} \begin{pmatrix} m_t & \slashed{P} \\ \slashed{P} & m_t \end{pmatrix} \begin{pmatrix} 0 & |2\rangle[3| \\ |3]\langle 2| & 0 \end{pmatrix} v(p_4), \tag{6.9}$$

which simplifies to,

$$-\frac{1}{\langle 2|p_4|3]\langle 23\rangle^2} \bar{u}(p_1) \begin{pmatrix} 0 & |3\rangle\langle 2|p_4|3][3| \\ |3]\langle 3|p_4|3]\langle 2| & m_t|3]\langle 32\rangle[3| \end{pmatrix} v(p_4). \tag{6.10}$$

If the result from the recursion relation is to match the result of the Feynman calculation this expression should be equivalent to the second diagram in figure 9. Making use of the Dirac equation one can simplify the Feynman calculation to

$$-\frac{1}{\langle 23\rangle^2} \bar{u}(p_1) \begin{pmatrix} 0 & \slashed{p}_3 \\ \slashed{p}_3 & 0 \end{pmatrix} v(p_4). \tag{6.11}$$

It may not be immediately obvious that the expressions (6.10) and (6.11) are equivalent, but they are. Firstly we note that the four component spinor can be written in terms of two component spinors,

$$u(p) = \begin{pmatrix} |u_p\rangle \\ |u_p] \end{pmatrix}. \tag{6.12}$$

This allows us to expand out (6.10) and immediately identify that the top row is in the correct form. The bottom row can be simplified by using $m_t|v_4] = \not{p}_4|v_4\rangle$ and by decomposing $|v_4\rangle = \alpha|3\rangle + \beta|2\rangle$ we can re-form the bottom row into the correct form and reconstruct (6.11).

This shows that recursion relations can be used to successfully calculate amplitudes with massive particles with spin on internal lines, as expected.

7. Conclusions

In this paper we have applied the recursion relations derived by Britto, Cachazo, Feng and Witten to a range of processes involving both internal and external, off-shell or massive particles. Our analysis includes massive particles with spin and this complements our earlier work [28] where we have considered massive scalars coupled to massless partons. As a first step, in section 4 we derived compact expressions for single vector boson currents where an off-shell gauge boson couples to a quark-antiquark pair and up to four gluons. These results both agree with and are more compact than the expressions previously available in the literature. New results for particular helicity configurations involving any number of partons were also derived.

Adding more off-shell vector bosons is also straightforward. The most direct approach is to divide the amplitude into its Abelian and Non-Abelian components. The Non-Abelian contribution is straightforwardly obtained by linking one or more currents with fewer vector bosons via triple or quartic vertices. On the other hand, the Abelian contribution is amenable to a recursive approach using the single Feynman diagram where all of the off-shell vector bosons couple to a fermion line in an ordered way. As an example, Sec. 5 contains compact expressions for the double vector boson current coupling to a quark-antiquark pair and up to two gluons in all helicity configurations.

Note that propagating particles with mass only show up in the Non-Abelian contribution and we were able to organise the recursive calculation of the Abelian part of multi-vector boson currents without having massive intermediate states. However, often it is necessary to deal with propagating massive particles and the recursion relations can handle this situation as well. The key point is to generalise the recursion relation to avoid using helicities for internal massive particles. The sum over helicities is then replaced by the conventional spin sum.

Taken together, the generalisations of the recursion relations presented here are capable of providing an efficient way of calculating tree-level amplitudes within the Standard Model and beyond.

Acknowledgments

A. Spinor conventions and useful idenitities

We use the standard $\sigma_{\mu\,\alpha\dot\alpha}$ and $\bar\sigma_\mu^{\dot\alpha\alpha}$ matrices to represent a vector p^μ in spinor notation

$$p_{\alpha\dot\alpha} = p^\mu \sigma_{\mu\,\alpha\dot\alpha}\,, \qquad p^{\dot\alpha\alpha} = p^\mu \bar\sigma_\mu^{\dot\alpha\alpha}\,. \tag{A.1}$$

Spinor indices are raised and lowered with ϵ-symbols, such that

$$\bar\sigma_\mu^{\dot\alpha\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma_{\mu\,\beta\dot\beta}. \tag{A.2}$$

We also have an identity $\bar\sigma_\mu^{\dot\alpha\alpha}\sigma^\mu_{\beta\dot\beta} = 2\delta^\alpha_\beta \delta^{\dot\alpha}_{\dot\beta}$.

Null vector p^μ_a is represented in terms of dotted and undotted 2-spinors as

$$p^{\dot\alpha\alpha}_a = a^{\dot\alpha} a^\alpha. \tag{A.3}$$

The spinor products are defined by

$$\langle ab\rangle = a^\alpha b_\alpha \equiv \langle a|^\alpha |b\rangle_\alpha, \tag{A.4}$$

$$[ab] = a_{\dot\alpha} b^{\dot\alpha} \equiv [a|_{\dot\alpha}|b]^{\dot\alpha}. \tag{A.5}$$

We note that the spinor summation conventions for dotted spinors in (A.5) follow what is usually used in QCD literature and differ by a minus sign from [49, 2, 3, 28].

Parity transformation requires complex conjugation which for spinors (of a real vector) is defined as

$$|a\rangle^*_\alpha = |a]_{\dot\alpha} \quad\Rightarrow\quad \langle ab\rangle^* = -[ab]. \tag{A.6}$$

The spinor sandwiches are defined in such a way that the summation over adjacent indices goes from up to down $^\alpha_\alpha$ for undotted indices and from down to up $_{\dot\alpha}{}^{\dot\alpha}$ for dotted indices, precisely as in (A.4) and (A.5)

$$\langle i|p|j] = \lambda_i^\alpha\, p_{\alpha\dot\alpha}\, \tilde\lambda_j^{\dot\alpha}, \tag{A.7}$$

$$\langle i|p_r p_s|j\rangle = \lambda_i^\alpha\, p_{r\,\alpha\dot\alpha}\, p_s^{\dot\alpha\beta}\, \lambda_{j\,\beta}, \tag{A.8}$$

$$[i|p_r p_s|j] = \tilde\lambda_{i\,\dot\alpha}\, p_r^{\dot\alpha\alpha}\, p_{s\,\alpha\dot\beta}\, \tilde\lambda_j^{\dot\beta}. \tag{A.9}$$

For massless momenta we have

$$s_{ij} = \langle i\,j\rangle[j\,i]\,, \qquad \langle i|p_r|j] = \langle i\,r\rangle[r\,j]\,, \tag{A.10}$$

$$\langle i|p_r p_s|j\rangle = \langle i\,r\rangle[r\,s]\langle s\,j\rangle\,, \qquad [i|p_r p_s|j] = [i\,r]\langle r\,s\rangle[s\,j]\,, \tag{A.11}$$

and so on.

Equation (A.7) also implies that

$$\langle a|\sigma_\mu|b] = \langle a|^\alpha \sigma_{\mu\,\alpha\dot\alpha}|b]^{\dot\alpha}\,, \quad [b|\bar\sigma_\mu|a\rangle = [b|_{\dot\alpha}\bar\sigma_\mu^{\dot\alpha\alpha}|a\rangle_\alpha \tag{A.12}$$

and the identity (A.2) gives

$$\langle a|\sigma_\mu|b] = [b|\bar\sigma_\mu|a\rangle, \tag{A.13}$$

or more generally

$$\langle a|\dots\sigma_\mu\dots|b] = [b|\dots\bar\sigma_\mu\dots|a\rangle, \tag{A.14}$$

$$\langle a|\dots\sigma_\mu\dots|b\rangle = -\langle b|\dots\bar\sigma_\mu\dots|a\rangle, \tag{A.15}$$

$$[a|\dots\sigma_\mu\dots|b] = -[b|\dots\bar\sigma_\mu\dots|a]. \tag{A.16}$$

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